HOMEWORK 5 COMPLEX ANALYSIS

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1. Problem 1

Consider the contour C in the complex plane consisting of a semicircle with radius R from $\theta = 0$ to $\pi/4$ in the complex plane. Then, we want to calculate

$$\int_C e^{iz^2} dz$$

We now break up our integral into 3 different parts. C_1 is the arc lying on the real axis, C_2 is the arc $Re^{i\theta}$, $\theta \in (0, \pi/4)$, and C_3 is the arc $re^{i\pi/4}$, $r \in (0, R)$.

It is clear that e^{iz^2} is holomorphic and thus by Cauchy's Theorem,

$$\int_C e^{iz^2} dz = 0$$

However, we also have:

$$\int_C e^{iz^2} dz = \int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz + \int_{C_3} e^{iz^2} dz$$

We now wish to calculate these three integrals to find our desired values. On C_1, clearly $z = x$, $dz = dx$, and $x \in (0, R)$. Thus,

$$\int_{C_1} e^{iz^2} dz = \int_0^R e^{ix^2} dx \to \int_0^\infty e^{ix^2} dx$$

As $R \to \infty$.

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For C_2, note that $z = Re^{i\theta}$, $\theta \in (0, \pi/4)$. Then,

$$\Big| \int_{C_2} e^{iz^2} dz \Big| \leq \int_{C_2} \Big| e^{iz^2} \Big| |dz| = \int_{C_2} \Big| e^{-R^2 \sin(2\theta)} \Big| |dz|$$

Using this, we can deduce that $e^{-R^2 \sin(2\theta)} \leq e^{-4R^2\theta/\pi}$, so that

$$\Big| \int_{C_2} e^{iz^2} dz \Big| \leq \frac{R\pi}{4} e^{-4R^2\theta/\pi} \to 0$$

As $R \to \infty$.

For C_3, we have that $z = re^{i\pi/4}$, $dz = e^{i\pi/4} dr$, with $r \in (0, R)$. Using this, and noting the direction of our integration,

$$\int_{C_3} e^{iz^2} dz = \int_R^0 e^{ir^2 (e^{i\pi/4})^2} e^{i\pi/4} dr = -\int_0^R e^{-r^2} e^{i\pi/4} dr$$

Where our final equality uses the fact that $e^{i\pi/2} = i$. Then, from here we let $R \to \infty$ and find that we must calculate $\int_0^\infty e^{-x^2} dx$. This is a famous integral, and can be calculated fairly easily by converting to polar coordinates and using the fact that this integral is separable. We have:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dy dx = \int_0^{2\pi} \int_0^\infty r e^{-r^2} dr$$

The integral on the right is a trivial substitution, $u = r^2$, so that $du/2 = r dr$, and we see:

$$\int_0^{2\pi} \int_0^\infty r e^{-r^2} dr = \pi$$

Also, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dy dx = \Big(\int_0^\infty e^{-x^2} dx \Big)^2$.

So that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Since the integrand is an even function, we immediately find:

$$\int_{0}^{\infty} e^{-r^2} dr = \sqrt{\pi}/2$$

Now, using the fact that $e^{i\pi/4} = \sqrt{2}/2 + i\sqrt{2}/2$, we see:

$$-\int_{0}^{R} e^{-r^2} e^{i\pi/4} dr = -\Big(\frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}\Big)$$

Combining the work for the above 3 contours, we have:

$$\begin{aligned} 0 &= \int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz + \int_{C_3} e^{iz^2} dz \\ &= \int_{0}^{\infty} e^{ix^2} dx - \Big(\frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}\Big) \\ \implies & \int_{0}^{\infty} e^{ix^2} dx = \Big(\frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4}\Big) \end{aligned} \tag{1.1}$$

By Euler's Formula, $\int_0^\infty e^{ix^2} dx = \int_0^\infty \cos(x^2) dx + i \int_0^\infty \sin(x^2) dx$, and comparing real and imaginary parts with the above, we find:

$$\int_{0}^{\infty} \cos(x^2) dx = \int_{0}^{\infty} \sin(x^2) dx = \frac{\sqrt{2\pi}}{4}$$

As desired.

2. Problem 2

Note that $f(z)$ as given in the problem statement is entire and as such its integral is independent of the contour γ. The proof of this is fairly trivial, since if you took two different arcs from one point to the other, the integral over these contours would be 0 using Cauchy's theorem. Separating these two integrals, we then see they coincide. Thus, we merely need to evaluate antiderivatives.

$$\int_\gamma (3z^2+5z+i)dz = z^3|_i^1 + 5z^2/2|_i^1 + iz|_i^1 = 1-i^3+5/2(1-i^2)+i(1-i)$$

Which simplifies to $7+2i$. (Note: If indeed you make the substitution $z = it + (1-t)$, with $t \in (0,1)$, we still find the same answer).

3. Problem 3

Again, by the logic of Problem 2, we merely need to evaulate at the antiderivative of e^z, which is obviously e^z. Then,

$$\int_\gamma e^z dz = e^z\Big|_1^{-1} = \frac{1}{e} - e$$

And we are done.

4. Problem 4

Using the definition of cosine,

$$\begin{aligned}\cos(z+\pi/2) &= \frac{e^{i(z+\pi/2)} + e^{-i(z+\pi/2)}}{2} \\ &= \frac{e^{i\pi/2}e^{iz} + e^{-i\pi/2}e^{-iz}}{2} \\ &= \frac{ie^{iz} - ie^{-iz}}{2} \\ &= -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z)\end{aligned} \tag{4.1}$$

Similarly,

$$\begin{aligned}
\cos(z+\pi) &= \frac{e^{i(z+\pi)} + e^{-i(z+\pi)}}{2} \\
&= \frac{e^{i\pi}e^{iz} + e^{-i\pi}e^{-iz}}{2} \\
&= \frac{-1e^{iz} - 1e^{-iz}}{2} \\
&= -\frac{e^{iz} + e^{-iz}}{2} = -\cos(z)
\end{aligned} \tag{4.2}$$

And,

$$\begin{aligned}
\cos(z+2\pi) &= \frac{e^{i(z+2\pi)} + e^{-i(z+2\pi)}}{2} \\
&= \frac{e^{2i\pi}e^{iz} + e^{-2i\pi}e^{-iz}}{2} \\
&= \frac{e^{iz} + e^{-iz}}{2} \\
&= \frac{e^{iz} - e^{-iz}}{2} = \cos(z)
\end{aligned} \tag{4.3}$$

Now, we can use the above to do the case for the sine function. Thus, $\sin(z+\pi/2) = -\cos(z+\pi) = \cos(z)$, so that

$$\sin(z+\pi/2) = \cos(z)$$

Similarly, note that

$$\sin(z+\pi) = -\cos(z+\pi+\pi/2) = \cos(z+\pi/2) = -\sin(z)$$

$$\implies \sin(z+\pi) = -\sin(z)$$

Finally,

$$\sin(z+2\pi) = -\cos(z+2\pi+\pi/2) = -\cos(z+\pi/2) = \sin(z)$$

$$\implies \sin(z+2\pi) = \sin(z)$$

and we are done.

5. Problem 5

Using the definitions for cosine and sine:

$$\begin{aligned} \cos(iy) &= \frac{e^{i(iy)} + e^{-i(iy)}}{2} \\ &= \frac{e^{-y} + e^{y}}{2} = \cosh(y) \end{aligned} \tag{5.1}$$

$$\begin{aligned} \sin(iy) &= \frac{e^{i(iy)} - e^{-i(iy)}}{2i} \\ &= -i\Big(\frac{e^{-y} - e^{y}}{2}\Big) = i\sinh(y) \end{aligned} \tag{5.2}$$

Using the above and employing the sum formulas for trig functions:

$$\begin{aligned} \cos(x+iy) &= \cos(x)\cos(iy) - \sin(x)\sin(iy) \\ &= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \end{aligned} \tag{5.3}$$

And similarly,

$$\begin{aligned} \sin(x+iy) &= \cos(x)\sin(iy) + \cos(iy)\sin(x) \\ &= \sin(x)\cosh(y) + i\cos(x)\sinh(y) \end{aligned} \tag{5.4}$$

For the complex forms of the above functions, it is natural to define

$$\begin{aligned} \cosh(z) &:= \frac{e^{z} + e^{-z}}{2} \\ \sinh(z) &:= \frac{e^{z} - e^{-z}}{2} \end{aligned}$$

Then, to differentiate merely recall the derivative of e^z (and the chain rule):

$$\begin{aligned} \big(\cosh(z)\big)' &= \frac{e^{z} + (-1)e^{-z}}{2} = \frac{e^{z} - e^{-z}}{2} = \sinh(z) \\ \big(\sinh(z)\big)' &= \frac{e^{z} - (-1)e^{-z}}{2} = \frac{e^{z} + e^{-z}}{2} = \cosh(z) \end{aligned}$$

And we are done.